Wireless Connectivity:
An Intuitive and Fundamental Guide

Chapter 6: A Mathematical View on a Communication Channel

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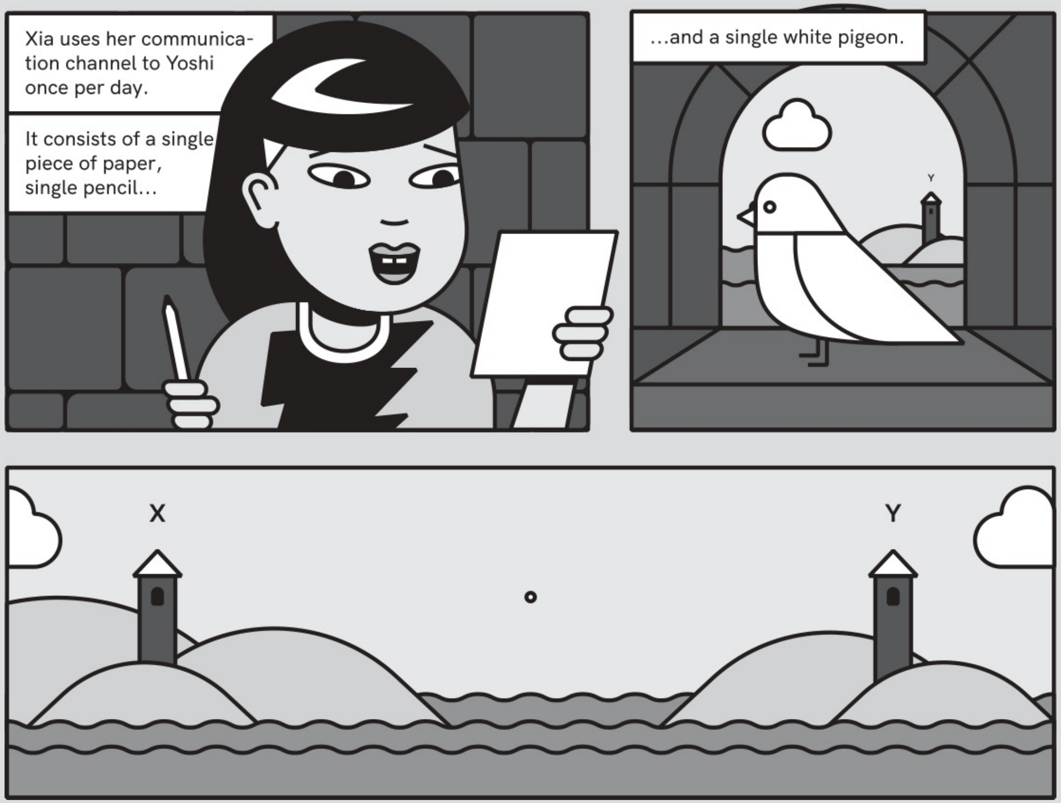
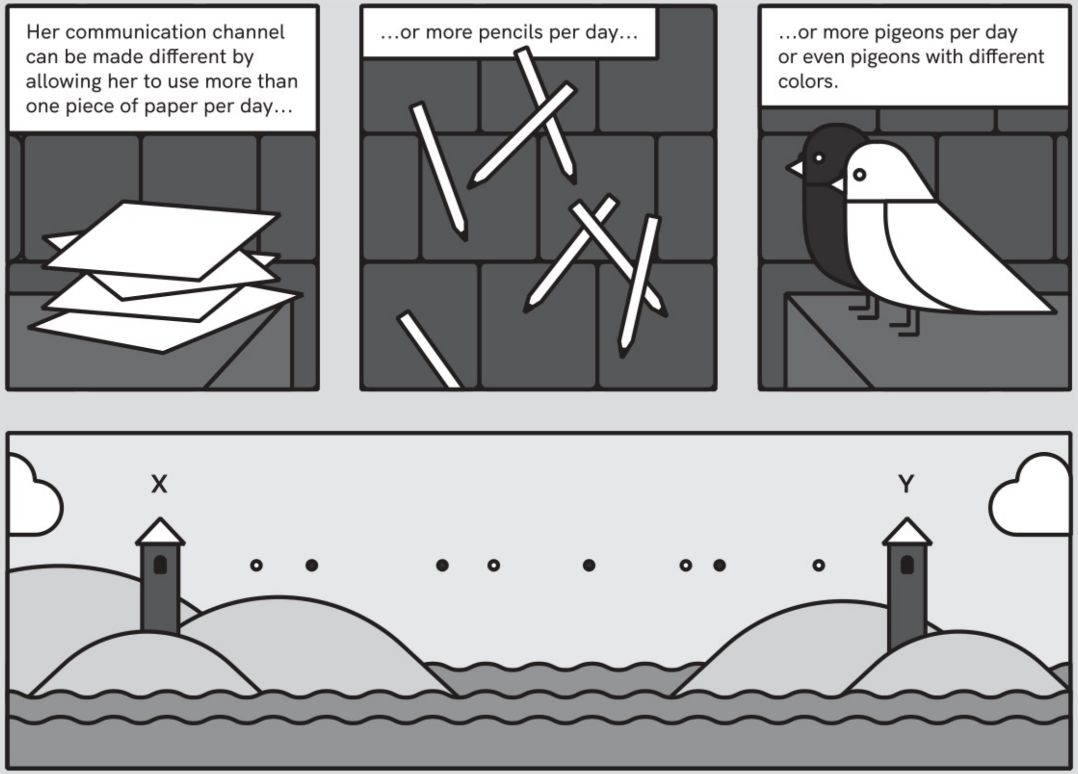
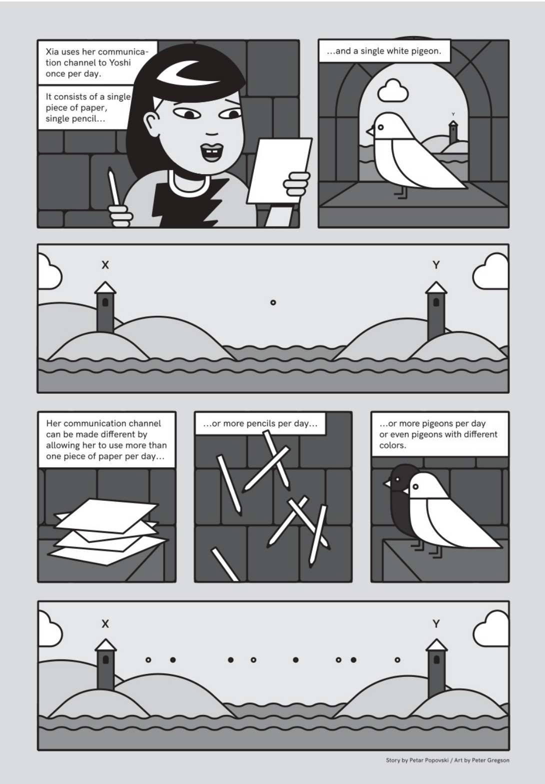
Modules

1. An easy introduction to the shared wireless medium
2. Random Access: How to Talk in Crowded Dark Room
3. Access Beyond the Collision Model
4. The Networking Cake: Layering and Slicing
5. Packets Under the Looking Glass: Symbols and Noise
6. **A Mathematical View on a Communication Channel**
7. Coding for Reliable Communication
8. Information-Theoretic View on Wireless Channel Capacity
9. Time and frequency in wireless communications
10. Space in wireless communications
11. Using Two, More, or a Massive Number of Antennas
12. Wireless Beyond a Link: Connections and Networks


Xia uses her communica-
tion channel to Yoshi
once per day.
It consists of a single
piece of paper,
single pencil...
...and a single white pigeon.
X
Y


Her communication channel
can be made different by
allowing her to use more than
one piece of paper per day...
...or more pencils per day...
...or more pigeons per day
or even pigeons with different
colors.
X
Y

Definition of a communication channel



- Channel consists of pencil, paper, pigeons, distance, and disturbances – all those are items that the sender will not or can not change

- Increasing the number of pigeons, pieces of paper, and pens increases the number of *channel uses*

What will be learned in this chapter

- What is a communication channel
- How the Additive white Gaussian noise (AWGN) channel varies with what knowledge is available to the actors
- How digital communication channels are built upon analogue channels
- How to obtain non-zero throughput

Mathematical view of the channel

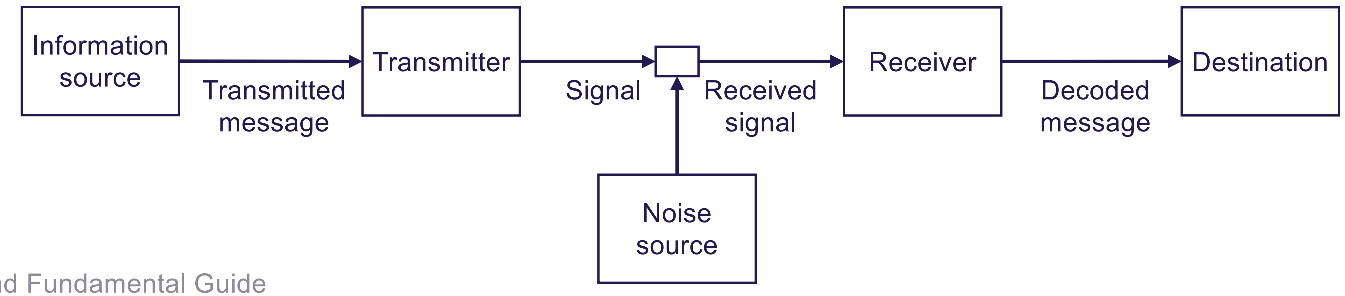
In the previous lecture, we introduced baseband models and focused on the noise

We take a more holistic view of the process which transforms the wireless signals and define **the channel:**

The channel is that part of the communication system that one is **"unwilling or unable to change"**

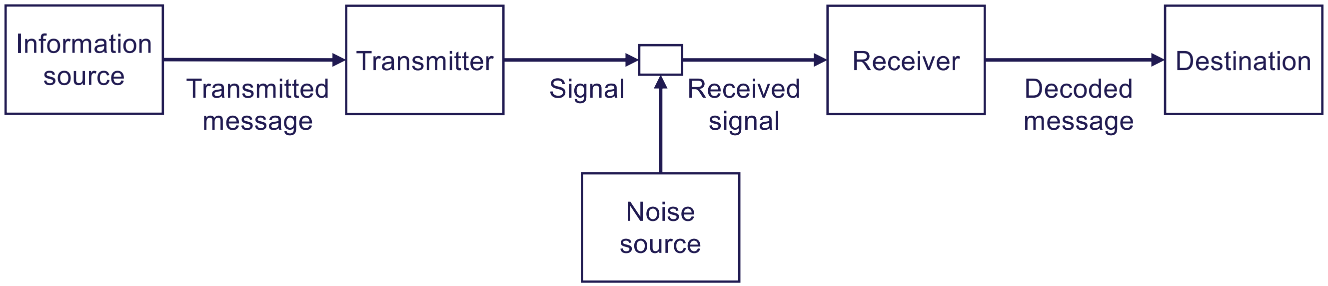
A ”black box” analogy to the layering concept

The **noise source** represents all the disturbances that can affect the transmitted signal



A simple channel model

The **noise source** represents all the disturbances that can make the received signal different from the transmitted signal



Toy example of the channel

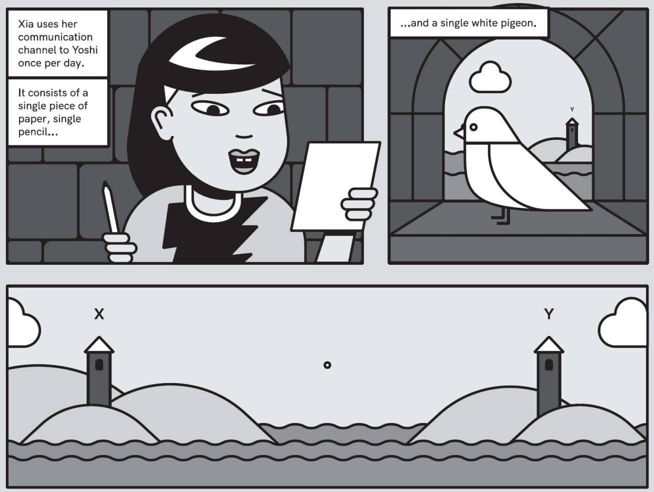
Xia sends a single letter to Yoshi every day

Xia has only **one** sheet of paper (space)

and **one** pencil (amount of written symbols) per day

The Pigeon, named Pigeon, flies once per day
and doesn't bring any information back.

What is the **data rate**?



The letter represents an atomic unit called *channel use*

Each channel use can carry a different amount of information

The letter can be subject to damage: rain

There is a probabilistic relationship between **input** and **output** of the channel

Specification of the channel

Formally:

$\mathcal{X}$ is a **finite** set of possible **inputs** x that Xia can transmit

- x is a whole possible content of the letter, not a single character

$\mathcal{Y}$ is a set of possible **outputs** received by Yoshi

- It is not necessarily identical to $\mathcal{X}$

The probabilities $p(y|x)$ for all possible pairs fully characterize the channel

- These are affected by noise

Other properties of the channel cannot be influenced by us, for example

- Better quality paper (antenna), higher number of pigeons, or their speed

Information carrying capacity

Let us define two more channels

1. **1/3 pen-channel:** 1 pencil given every third day

Clearly, the total amount of information is lower

2. **3 pen-channel:** 3 pencils given every third day

The 1 pen-channel is only a **special case that we can relate to power**

1 pen-channel imposes maximum power P per channel use.

- Exemplary sequence for six days: $[P,\ 0.8P,\ P,\ 0.7P,\ 0.5P,\ P]$

3 pen-channel imposes maximum power $3P$ over three channel uses

- **More possibilities** $[2P,\ P,\ 0,\ 1.8P,\ 0.5P,\ 0.7P]$

Analog channels with Gaussian noise

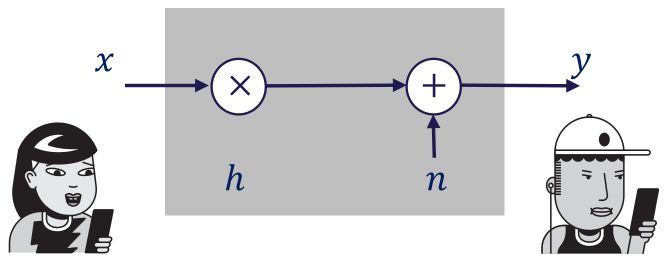
When **both input** x and **output** y are continuous (real or complex) numbers we have an **analog channel**:

$$\boldsymbol{y} = h\boldsymbol{x} + \boldsymbol{n}$$

The AWGN channel is a **special case,** where:

- h is **known and constant** during the transmission of a packet
- n is a Gaussian random variable with zero-mean and variance σ^2
- Xia's symbols fulfill power constraint

$$\frac{1}{L}\sum_{i=1}^{L}|x_i|^2 \leq P$$



Memoryless channel: y_i depends only on x_i and n_i

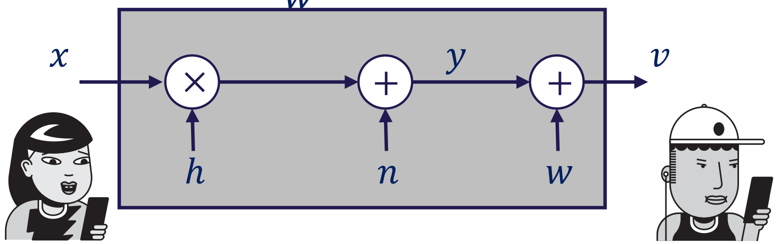
Analog channels with Gaussian noise

We usually assume y is observed perfectly

If not, we can consider a different channel where y is observed through v, specified by $p(v|y)$. This is a **cascade** channel

- **If** $v = y + w = hx + n + w$ and both n and w are Gaussian, we can simply consider an equivalent channel with noise variance $\sigma^2 + \sigma_w^2$

Xia can use any specific modulation that fulfills the power constraint



But, by using the same modulation constellation with power P for each transmitted symbol, we're using the Gaussian channel in a suboptimal way

L symbols can be seen as a super-symbol as in the 3-pen channel

Who knows what about the channel?

The **channel coefficient** h summarizes
the effects of propagation environment, antennas, etc.

- Its changes heavily affect the quality of communication
- **Crucial** to understand the impact of the (lack of) **knowledge** of h

Example

At each transmission

- $h = 0.5$ (with probability $p = 0.625$)
- $h = -1.5$ (with probability $p = 0.375$)

Channel use no.	1	2	3	4	5
Channel state h	0.5	0.5	-1.5	0.5	-1.5

Xia uses BPSK and transmits -1 to denote bit value 0 and 1 to denote bit value 1

Who knows what about the channel?

Example (cont.)

Case 1. AWGN channel: Both Xia and Yoshi **know** *h*

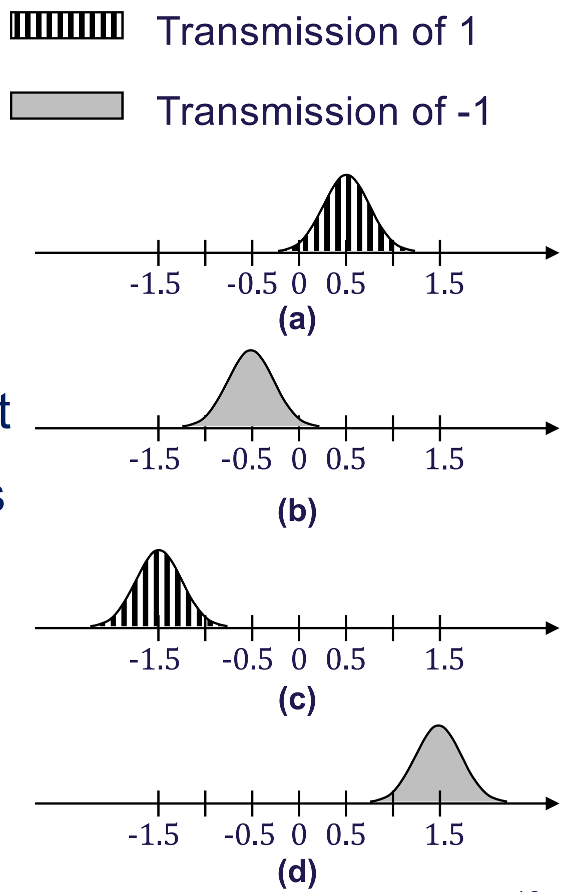
If $h = 0.5$, Yoshi observes y as in Fig. **(a)** and **(b)**

Else if $h = -1.5$, Yoshi observes y as in Fig. **(c)** and **(d)**

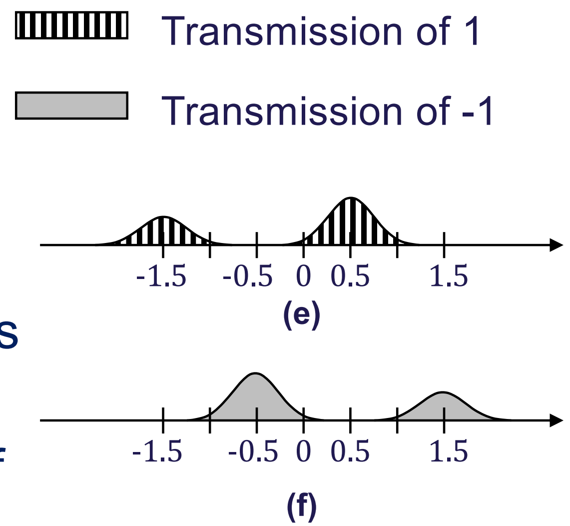
- Transmitting 1 results in $y < 0$, but Yoshi **is aware** of that

Xia can take **advantage of her knowledge** of the channels and apply adaptive modulation e.g.

- When channel is **weak**,
  use more robust QPSK modulation
- When channel is **strong**, use 8-PSK modulation



Who knows what about the channel?



Example (cont.)

Case 2. Neither Xia nor Yoshi **know** the channel

Now h is another type of noise (multiplicative)

Fig. **(e)** and **(f)** show the distribution of y when Xia transmits 1 and -1, respectively

- Notice, the height differences of the curves as a result of the **probability distribution** of h

If Xia **knows the distribution** of h
(but not the instantaneous value)
then she could use different symbols

For example, $\{0, \sqrt{2}\}$ instead of $\{-1,1\}$

Who knows what about the channel?

Example (cont.)

Case 3. Xia **knows** the channel, but Yoshi doesn't

If Xia **does nothing to help** Yoshi, he will again observe the distributions **(e)** and **(f)**

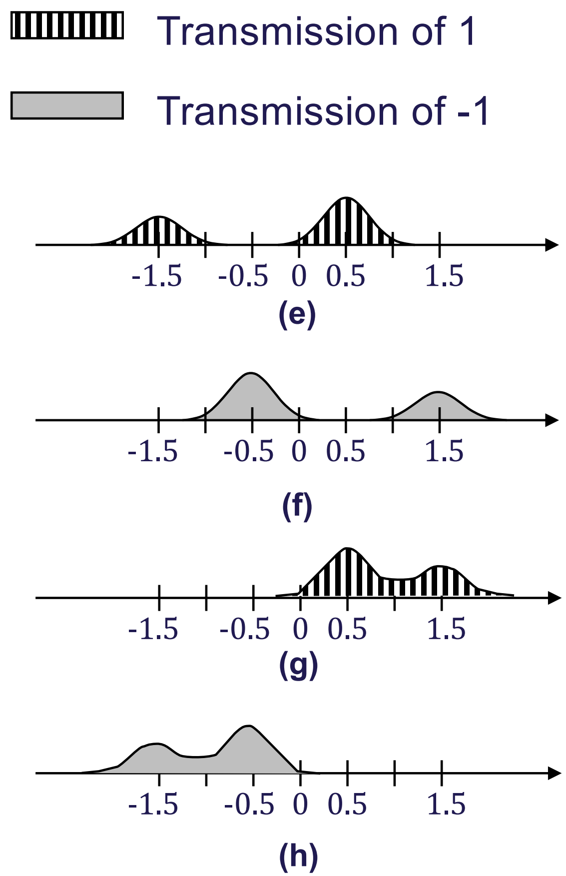
Since Xia **knows** h, she can control the distribution of the signal arriving at Yoshi

(g) Xia wants to transmit 1 so that Yoshi observes $y > 0$

If $h = 0.5$ she transmits 1. **Else if** $h = -1.5$, then -1

(h) Xia wants to transmit 0 so that Yoshi observes $y < 0$

If $h = 0.5$ she transmits -1. **Else if** $h = -1.5$, then 1



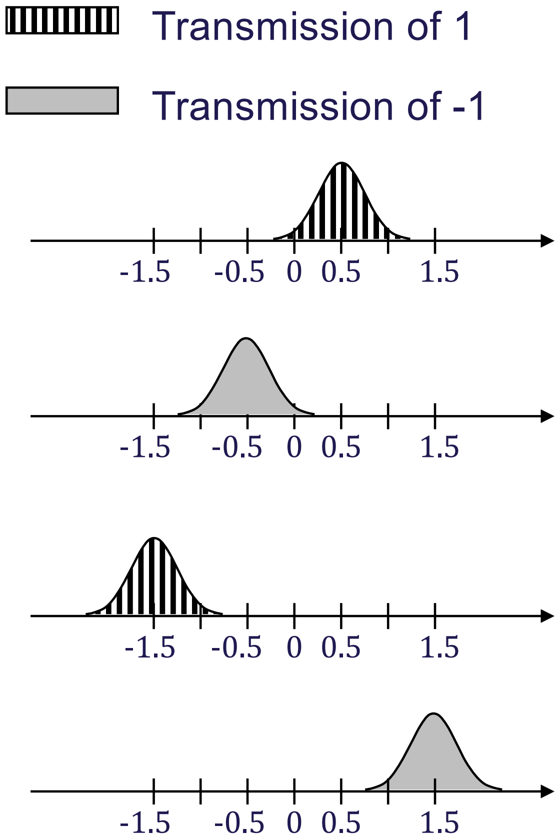
Who knows what about the channel?

Example (cont.)

Case 4. Only Yoshi **knowns** the channel

The decision is simple: Yoshi who decides what was sent by looking at **the sign** of y

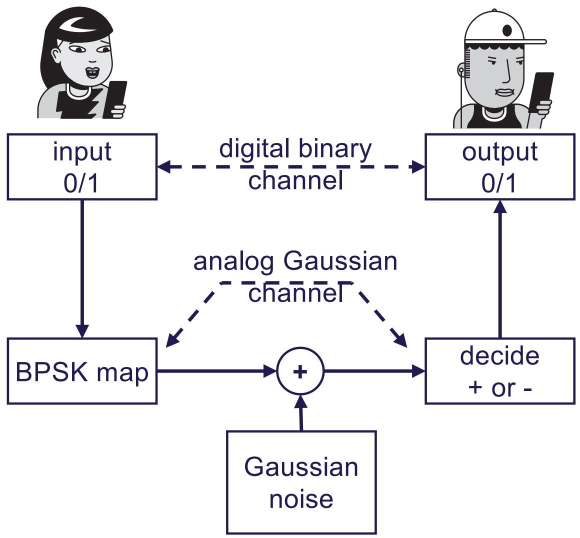
But Xia **cannot** use **adaptive modulation** to exploit **strong** and **weak** channel coefficients

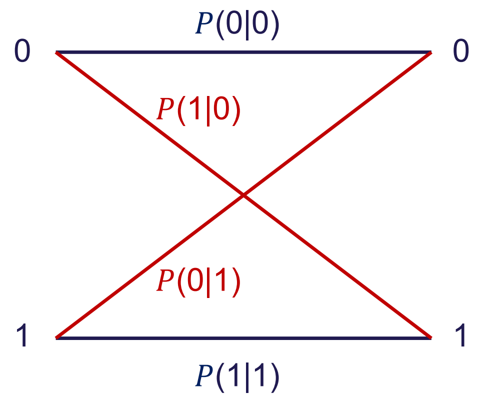


Using analog to create digital channels

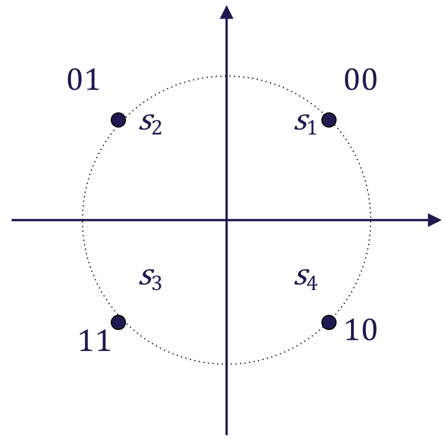
Simplest binary channel: BPSK over AWGN channel

- The module maps $0 \rightarrow -\sqrt{P}, \;\; 1 \rightarrow \sqrt{P}$
- The complementary module on the receiver performs matched filtering
$$|h|^2 x + h^* n$$
and decides what was transmitted by inspecting the **sign**
- The probabilities of **error** $P(1|0)$ and $P(0|1)$ are determined by **power** P and **noise** variance σ^2





Using analog to create digital channels

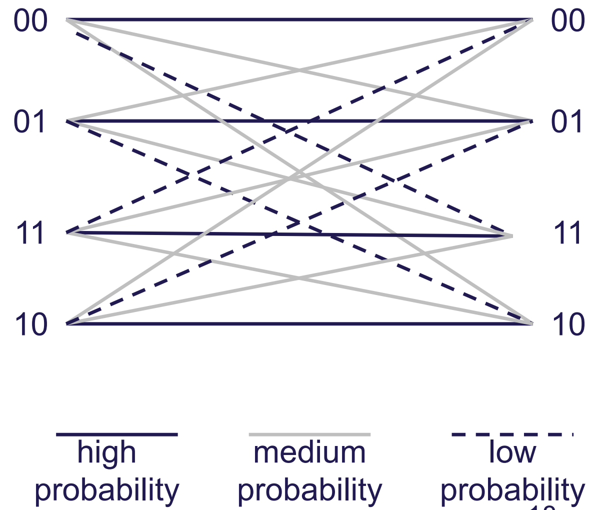


Another type of digital channel:
QPSK over AWGN channel

Xia groups bits **by two**, and the module maps them onto points from $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$

The **input** and **output** correspond to two-bit vectors

$$\boldsymbol{x}_i, \boldsymbol{y}_i \in \mathcal{B} = \{00,01,10,11\}$$



The channel is fully specified by conditional probabilities $P(\boldsymbol{y}_i|\boldsymbol{x}_i)$. But different mapping of bit vectors to symbols **changes** the channel

From the properties of the Gaussian noise:

$$P(00|00) > P(01|00) = P(10|00) > P(11|00)$$

Gray mapping: $P(y_1 y_2|x_1 x_2) = P(y_1|x_1) \times P(y_2|x_2)$

Using analog to create digital channels

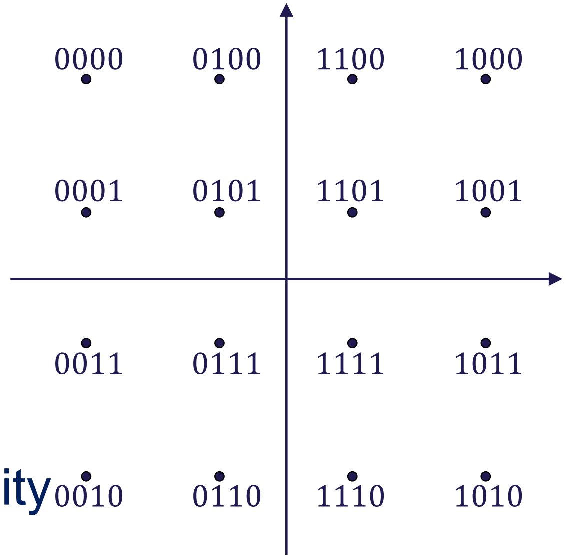
Higher-order constellations: 16-QAM over AWGN

Gray mapping is only a local property

Unlike in BPSK and QPSK,
some symbols have different **error** probabilities

Example: If $Re\{n\} > 0$

- **If** $\boldsymbol{x} = (1100)$**,** it can be mistaken as (1000)
- **Else if** $\boldsymbol{x} = (1000)$, shift to the right doesn't create ambiguity



Bits at different positions have different **error** probabilities

- Therefore $P(y_1 y_2 y_3 y_4 | x_1 x_2 x_3 x_4) \neq P(y_1|x_1) \times P(y_2|x_2) \times P(y_3|x_3) \times P(y_4|x_4)$

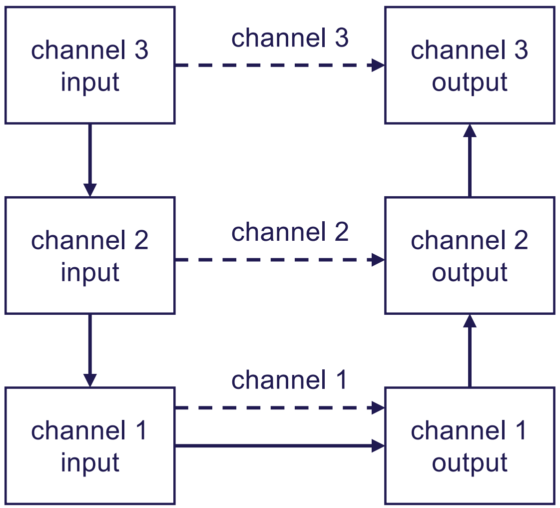
Note: Sending one 16-QAM symbol is **not equivalent** to four BPSK transmissions (in terms of **error**)

Transmission of packets

We've seen that channels exhibit hierarchical structure

A possible way to construct a channel:

- **Channel 1** is a binary symmetric channel (BSC) with inputs/outputs $\{0,1\}$
- **Channel 2** has **three** inputs/outputs $\{A, B, C\}$
  - $A \rightarrow 0,\quad B \rightarrow 10,\quad C \rightarrow 11$
- **Channel 3** has **eight** inputs/outputs $\{Z_1, Z_2, \dots, Z_8\}$
  - $Z_1 \rightarrow AA,\quad Z_2 \rightarrow AB,\quad Z_3 \rightarrow AC,\quad Z_4 \rightarrow BA,$ $Z_5 \rightarrow BB,\quad Z_6 \rightarrow BC,\quad Z_7 \rightarrow CA,\quad Z_8 \rightarrow CB$



But such a design might not be efficient in terms of resources. In general, we can:

Define a **higher layer (HL)** channel operating on packets of length L based on a **lower layer (LL)** channel having S inputs/outputs, with S^L inputs/outputs

Transmission of packets

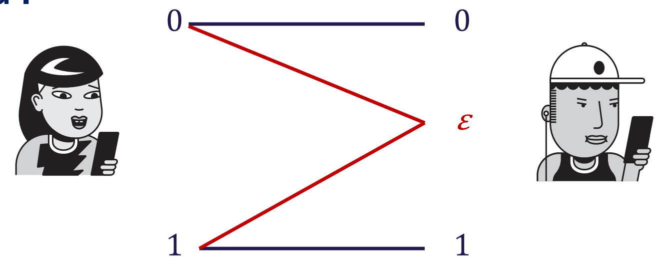
Bundling multiple channel uses is essential to provide reliability

Scenario: Xia sends a single bit, and with probability p it will be flipped

Yoshi receives 0 or 1 but has no way to verify which one was sent

The throughput is 0!

What if the channel is a **binary erasure channel** instead?



If Yoshi receives 0 or 1: this is valid data;

Else Yoshi asks Xia to retransmit

Now the throughput is $(1 - p)$

The erasure functionality is achieved by dedicating extra bits/symbols

A **reliable** high-level (HL)Channel carrying b data symbols can be constructed from $b + c$ channel uses of a low-level (LL)Channel.

Transmission of packets

The new HLChannel has 2^b possible **inputs** $\omega_1, \dots, \omega_{2^b}$ and 2^{b+c} possible **outputs**

The extra $2^{b+c} - 2^b$ values result in **erasure**

Ideally, a transmitted symbol ω_i is received as ω_i or an erasure

- No **undetected errors**: **output** is not equal to any valid ω_j s.t. $\omega_j \neq \omega_i$

But, this is **impossible to ensure in practice**: when finite $b + c$ bits are sent, any of the 2^{b+c} outputs has nonzero probability

Formally, denote by P_d and P_u the **detectable and undetectable error probability**.

Erasure probability: $P_e = 1 - (1 - p)^{b+c} = P_d + P_u \approx P_d$

***Nominal data* rate:** $R = \frac{b}{b+c} \left[\frac{\text{bits}}{\text{channel use}}\right]$ ($R \log_2 S$ with higher modulation)

Goodput: $G = R(1 - P_e) \left[\frac{\text{bits}}{\text{channel use}}\right]$

Transmission of packets

So far, we have assumed that Yoshi **knows if and when** Xia will transmit

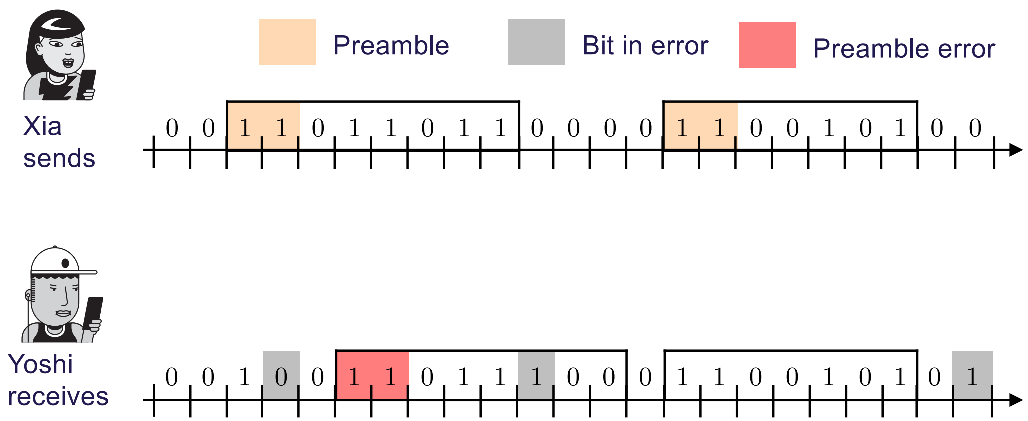
What if Xia transmits “nothing”? What is nothing?

There were cases where “nothing” was also a valid symbol denoting bit value 0
Example: Communication with asymmetric power levels $\{0, \sqrt{2}\}$

In this case “0” has a **dual role:** represents the idle state and bit value 0

How to tell the difference?

- Add r symbols called **preamble** at the beginning of the packet



This problem is known as **frame synchronization**

Transmission of packets

What happens if we define separate symbols for 0 and idle?

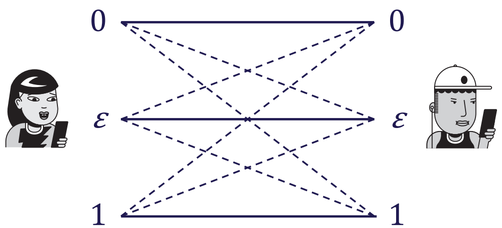
Example: BPSK with **ternary input** $\{0, \epsilon, 1\}$ corresponding to analog symbols $\{-1,0,1\}$

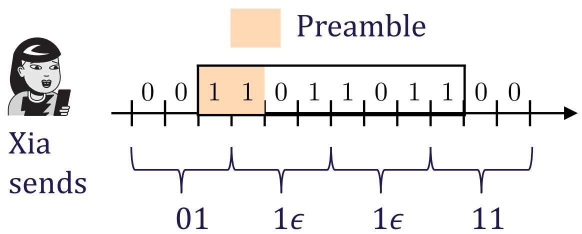
Synchronization seems easier now that Xia can control the idle signal

- However, due to noise, a valid symbol can be interpreted as ϵ mid-packet!

More importantly, this is **not** the information-theoretic way of thinking:

- **If** Xia can produce another symbol, she should use it to carry data and let the **preamble** take care of synchronization





Bit combination	Transmitted symbols
000	00
001	01
010	0ϵ
011	10
100	11
101	1ϵ
110	$\epsilon 0$
111	$\epsilon 1$

Outlook and takeaways

- **The channel** is that part of the communication system that one is **“unwilling or unable to change”.**
- Channel knowledge changes communication scheme
- Error detection necessary for non-zero throughput
- Digital channels built upon analog channels
- Data networking versus information-theoretic views.