Wireless Connectivity:
An Intuitive and Fundamental Guide

Chapter 9: Time and Frequency in Wireless Communications

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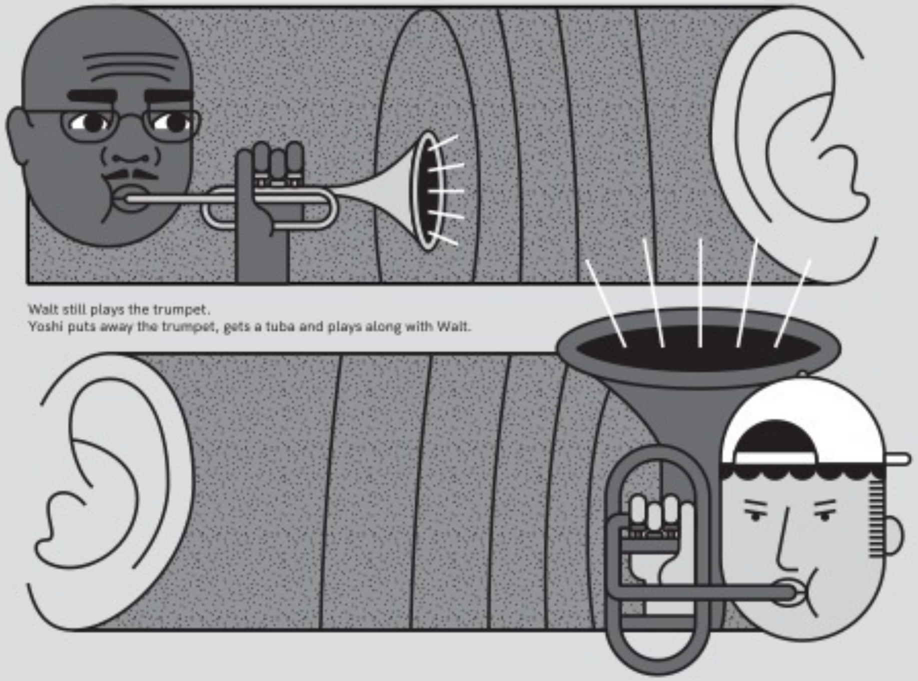
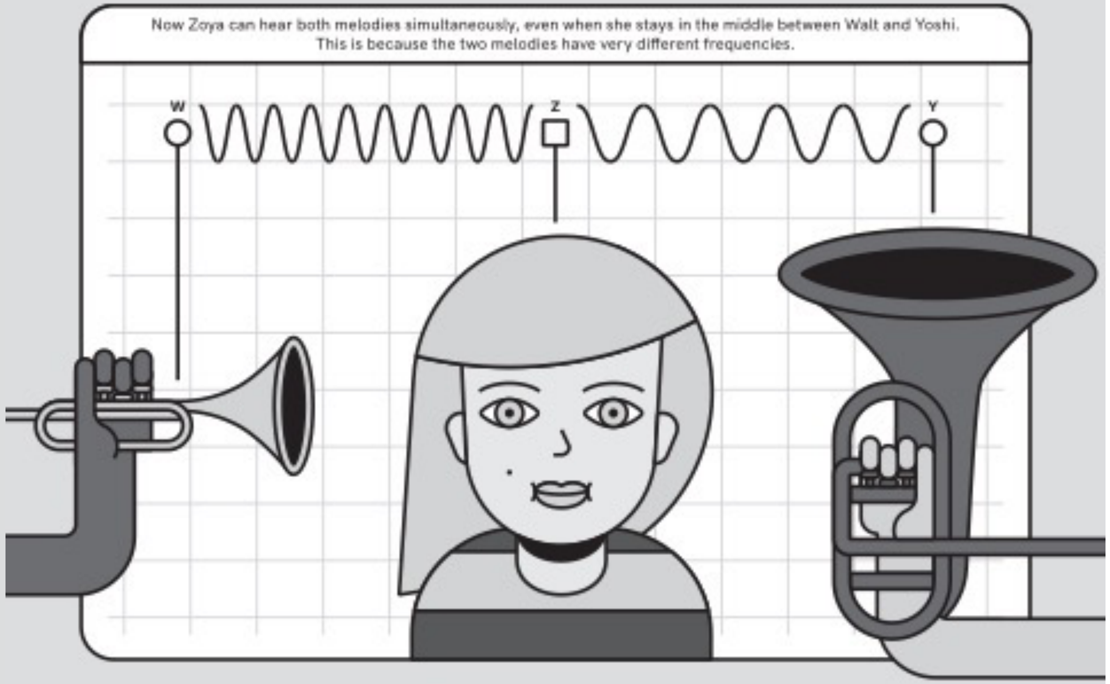
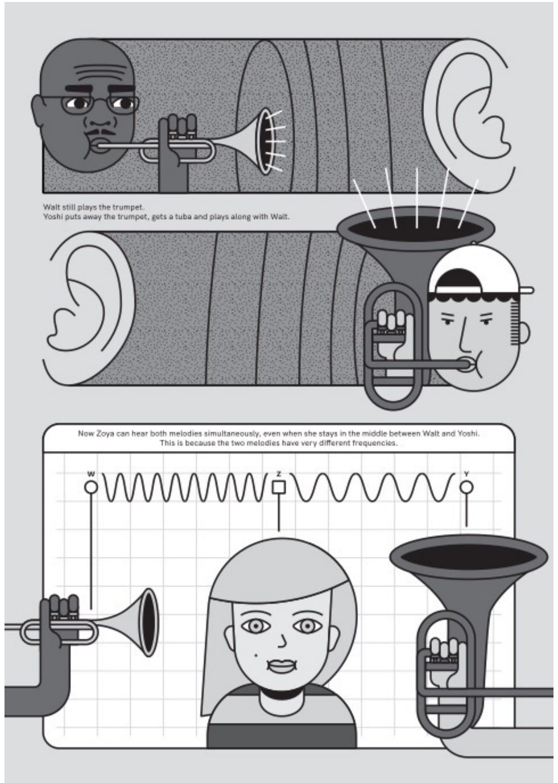
Modules

1. An easy introduction to the shared wireless medium
2. Random Access: How to Talk in Crowded Dark Room
3. Access Beyond the Collision Model
4. The Networking Cake: Layering and Slicing
5. Packets Under the Looking Glass: Symbols and Noise
6. A Mathematical View on a Communication Channel
7. Coding for Reliable Communication
8. Information-Theoretic View on Wireless Channel Capacity
9. **Time and Frequency in Wireless Communications**
10. Space in Wireless Communications
11. Using Two, More, or a Massive Number of Antennas
12. Wireless Beyond a Link: Connections and Networks


Walt still plays the trumpet.
Yoshi puts away the trumpet, gets a tuba and plays along with Walt.


Now Zoya can hear both melodies simultaneously, even when she stays in the middle between Walt and Yoshi.
This is because the two melodies have very different frequencies.
W
Z
Y

Time and frequency



- Signals are eventually taking place in the physical world and consume time
- Different properties of signals in time that allow signal separation and multiplexing: frequencies
- Frequency channels and multi-user communication

What will be learned in this chapter

- Relation between discrete communication symbols and analog waveforms
- Degrees of freedom, intersymbol interference (ISI)
- Frequency in waveforms, orthogonality and bandwidth limitation
- Power spectrum and capacity of bandlimited channels
- Multiple access, duplexing, and spread spectrum

Transmission of discrete values

Medium access control (MAC) protocols operate with notion of **discrete time**

The main entity at a MAC layer is a packet

Reliable communication is a process of sending discrete values

Physical signals and **waveforms** are **analog** (**continuous time** signals)

Distinguishing M waveforms means sending $\log_2 M$ bits of information

Receiver identifies the **class of waveform** received

Considering **physical** channels that consume time:

- How many bits per second [bps] can be **reliably** communicated by using **analog signals**?

Degrees of Freedom in communication

To find the rate R, in bits per second, we need to find:

1. The number of channel uses per second D, called **degrees of freedom (DoFs)**
2. The channel capacity C in bits per channel use $[\text{bits/c.u.}]$

$$R = D \cdot C \quad [\text{bps}]$$

DoFs **bridge** the concepts of
abstract communication channels and **analog waveforms**

Sampling **continuous waveforms** creates a discrete set of channel uses per time

- We know how to communicate over discrete channel uses

*NB: With DoF we mean **real DoF;** a complex DoF has 2 real DoFs.*

D basis waveforms become a linear combination controlled by the real coefficients

A simple conversion of a symbol into a waveform

Example: The TXmodule of Zoya is a black box.

Produces **a continuous waveform** as an output that is scaled with a **real number** (input)

Pulses are $T = 1.5$ seconds apart

The inputs are $z = \{-3, -1, 1, 3\}$ and represent $\{00, 01, 10, 11\}$

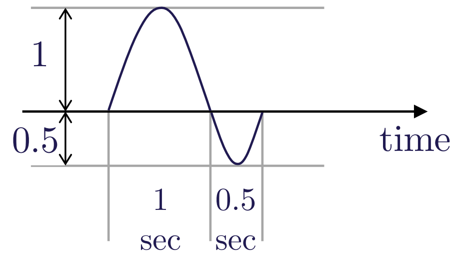
The output signal is $y = z + n$, where n is the noise

For a channel capacity C, we have

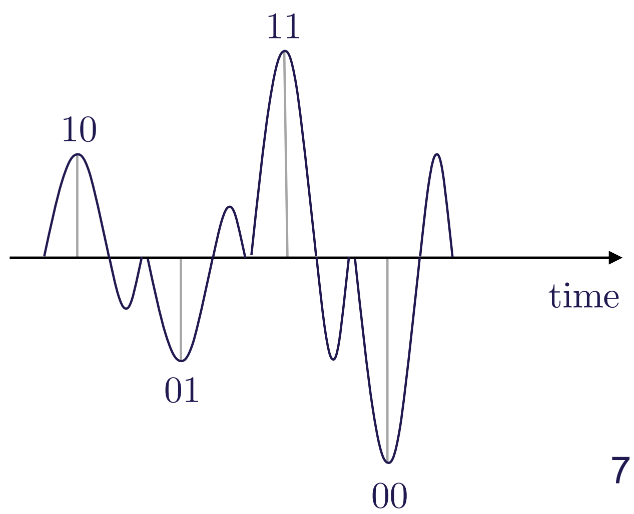
$$R = \frac{C}{T} \quad [\text{bps}]$$

Reducing T may increase the rate R

Elementary waveform



Scaled waveform



Overlapping waveforms and intersymbol interference (ISI)

Example (cont.)

Pulses are $T = 1.5$ seconds apart.

For a channel capacity C, we have

$$R = \frac{C}{T} \quad \text{[bps]}$$

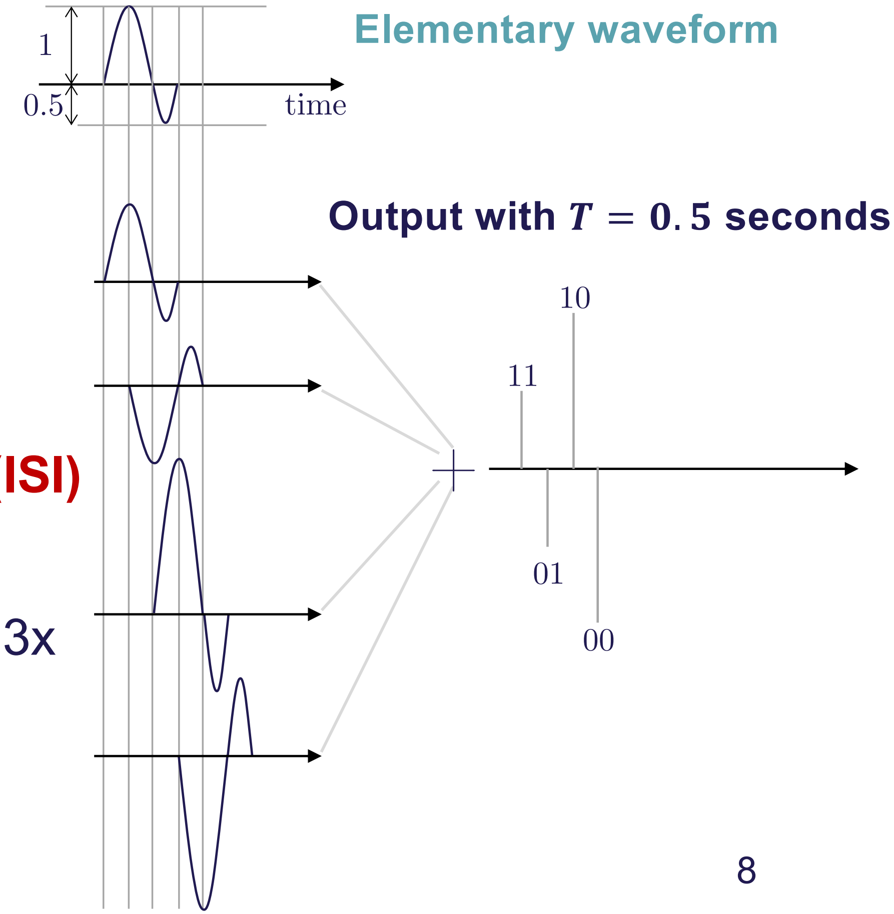
Reducing T may increase the rate R

But also, may create **inter-symbol interference (ISI)**

ISI removes the **memoryless property**

By setting $T = 0.5$ there is no **ISI** and R increases 3x

Can T become arbitrarily low?



Frequency characteristics of waveforms: I and Q components

Increasing the data rate R by compressing the **waveform** in time

✓ More channel uses per second

✗ Implies faster changes and sampling of the circuits

Amplitude and phase representation: $\tilde{z}_f = |A| \cos(2\pi f t + \phi)$

If frequency f is fixed: Amplitude $|A|$ and phase ϕ are the two DoFs.

In-phase (I) and quadrature (Q) representation: $\tilde{z}_f = z_{I,f} \cos(2\pi f t) - z_{Q,f} \sin(2\pi f t)$

Orthogonality: the receiver can extract $z_{I,f}$ and $z_{Q,f}$ independently

$z_{I,f} = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{z}_f(t) \cos(2\pi f t) \mathrm{d}t$ and $z_{Q,f} = -\frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{z}_f(t) \sin(2\pi f t) \mathrm{d}t$ are orthogonal on $\left[-\frac{T}{2}, \frac{T}{2}\right)$

Orthogonality with multiple frequencies

How many independent, non-interfering DoFs, can Yoshi extract from the signal sent by Zoya?

Extending the **orthogonality** to signals with **different frequencies**

Orthogonality interval of length T:

Any sinusoidal of frequency $\frac{k}{T} = kf$ is orthogonal to those of frequency νf if $k \neq \nu$

$$\tilde{z}(t) = \sum_{k=0}^{\infty} \tilde{z}_{k,f}(t) = \sum_{k=0}^{\infty} z_{I,kf} \cos(2\pi k f t) - z_{Q,kf} \sin(2\pi k f)$$

By choosing all $z_{I,kf}$ and $z_{Q,kf}$, Zoya **synthetizes** a periodic **waveform**

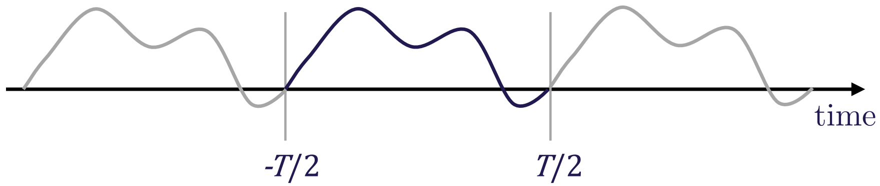
Yoshi can receive an infinite amount of symbols $\{z_{I,kf}, z_{Q,kf}\}$ through **Fourier analysis**

Bandwidth and time-limited signals

There is an infinite number of orthogonal **sinusoidal waveforms** within T, right?

Fourier theory says: **yes**. However, the **energy** E **is finite**

$$E = \sum_{k=0}^{\infty} E_{kf}; \qquad E_{k,f} = \left(z_{I,kf}^2 + z_{Q,kf}^2\right)T$$



How? Limit $E_{k,f} = 0$, for all $k \geq k_h$. This is a band-limited signal: $[f_L, f_H]$

A bandwidth of W [Hz] (including $f_L = 0$) contains at most

$$DoF_{\text{lowpass}} = \frac{2W}{f} + 1 = 2WT + 1$$ coefficients, or

$$DoF_{\text{bandpass}} = \frac{2W}{f} = 2WT$$ coefficients

How can we keep the energy finite while having infinite number of frequencies?

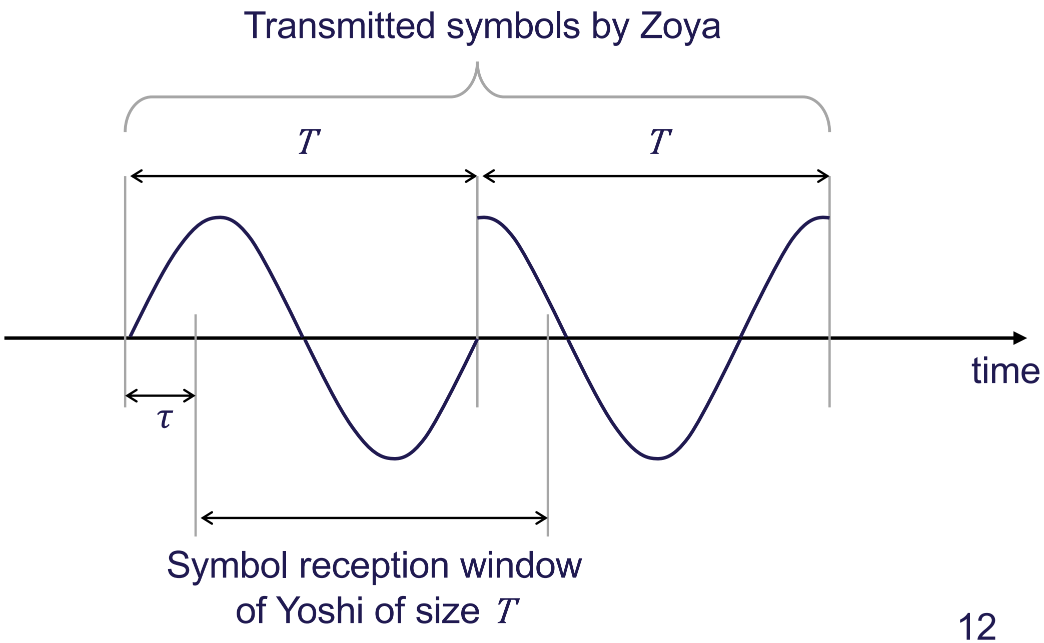
How to have a limited bandwidth and still consider signals in limited time?

We need to look from the receiver perspective!

In each T interval: Zoya modulates symbols $z_{I,kf}(i), z_{Q,kf}(i)$

Requires perfect synchronization

Otherwise, there is **ISI**



Parallel communication channels

This is one of the main **benefits** of introducing **frequency:**

Fix T and W → obtain $2WT$ **parallel symbols**

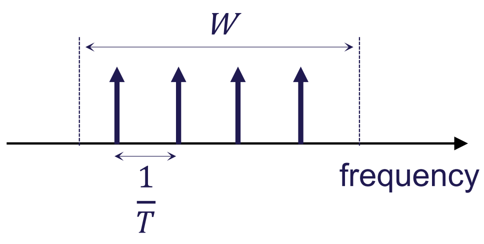
Subcarriers are the basis for orthogonal frequency division multiplexing **(OFDM)**

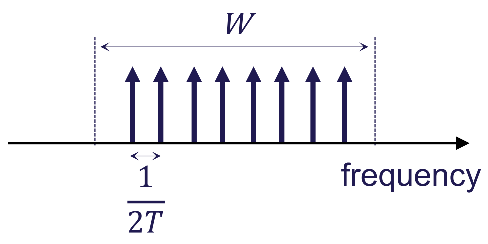
Assume: Each subcarrier is modulated with M-ary symbols

Each symbol carries $\log_2 M$ bits, then

$$R = \frac{2WT \log_2 M}{T} = 2W \log_2 M \text{ [bps]}$$

The rate R is **independent** on T





How frequency affects multiple access

If T is agreed, the bandwidth W can be **split between users in the uplink**

- This is known as **channelization**
- Frequency division multiple access **(FDMA)**
- **If** subcarriers are **orthogonal**, then we have orthogonal FDMA **(OFDMA)**

TDMA and **FDMA** have the **same number of DoFs and capacity**

So far, there is no difference in the two, but…

What if the **different users** use **different systems**?

Splitting the bandwidth allows for system coexistence.

If the bandwidth is split, each system can define its own synchronization reference

Signal power and Gaussian noise

Power over a time duration: $P_z = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{z}^2(t)\mathrm{d}t$

Overall signal power: $P_z = \sum_{k=1}^{\infty} z_{I,kf}^2 + z_{Q,kf}^2 = \sum_{k=1}^{\infty} P_{z,kf}$

Total power at frequency $kf = k/T$: $P_{z,kf}$

Signal at Yoshi (in-phase): $y_{I,kf} = z_{I,kf} + n_{I,kf}$; s.t. $n_{I,kf} = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{n}(t)\cos(2\pi kft)\,\mathrm{d}t$

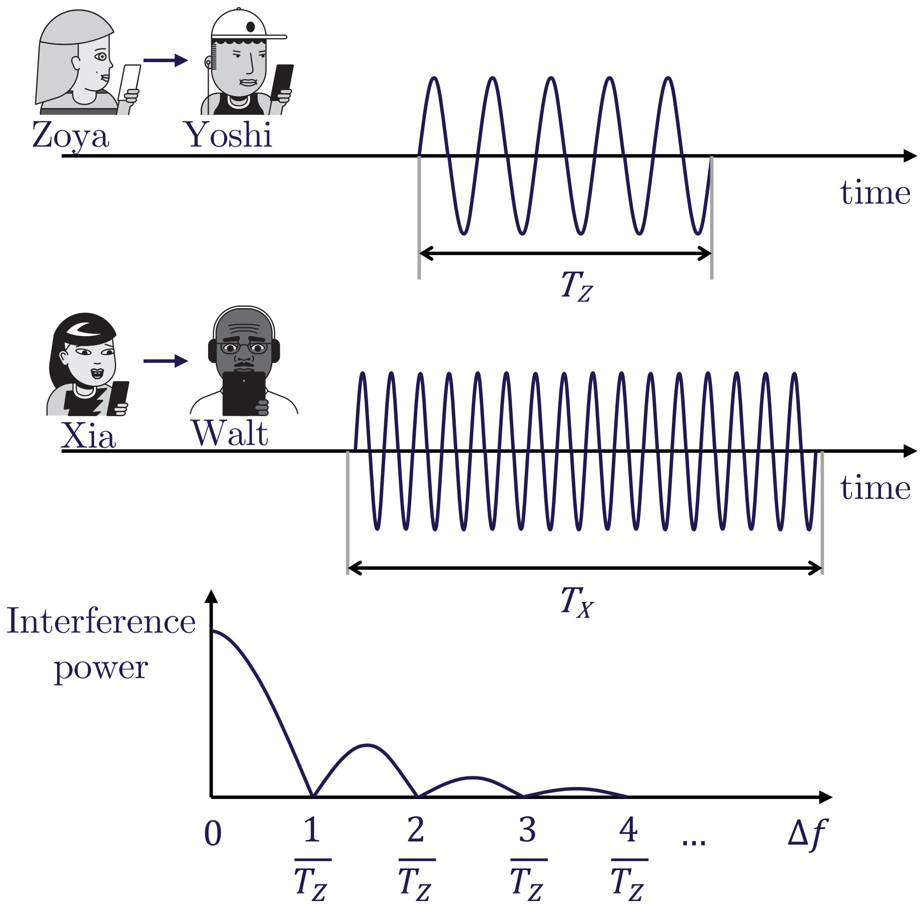
A**W**GN $\tilde{n}(t)$ affects all frequencies, s.t. $E\left[n_{I,kf}^2\right] = E\left[n_{Q,kf}^2\right] = \frac{P_N}{2}$, for each kf

Propagation distortion per subcarrier: $y_{kf} = h_{kf} z_{kf} + n_{kf}$;

Then, the SNR is $\gamma_{kf} = \frac{\left|h_{kf}\right|^2 P_{z,kf}}{P_N}$

Interference between non-orthogonal frequencies

Zoya communicates with Yoshi
AND
Xia communicates with Walt



Yoshi receives $\tilde{y}(t) = \tilde{z}(t) + \tilde{x}(t)$

Useful signal $\tilde{z}(t) = \cos(\omega_1 t)$

Interference $\tilde{x}(t) = \cos((\omega_1 + \Delta\omega)\, t + \phi)$

$y = \frac{2}{T_1}\int_0^{T_1} \tilde{y}(t)\cos(\omega_1 t)\mathrm{d}t = 1 + x$, where

$$x = \frac{2}{T_1}\int_0^{T_1} \cos(\omega_1 t)\cos\big((\omega_1 + \Delta\omega)t + \phi\big)\mathrm{d}t$$

Sufficient separation is required for interference to be negligible

Power spectrum and Fourier transform

Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t$$

Parseval's theorem

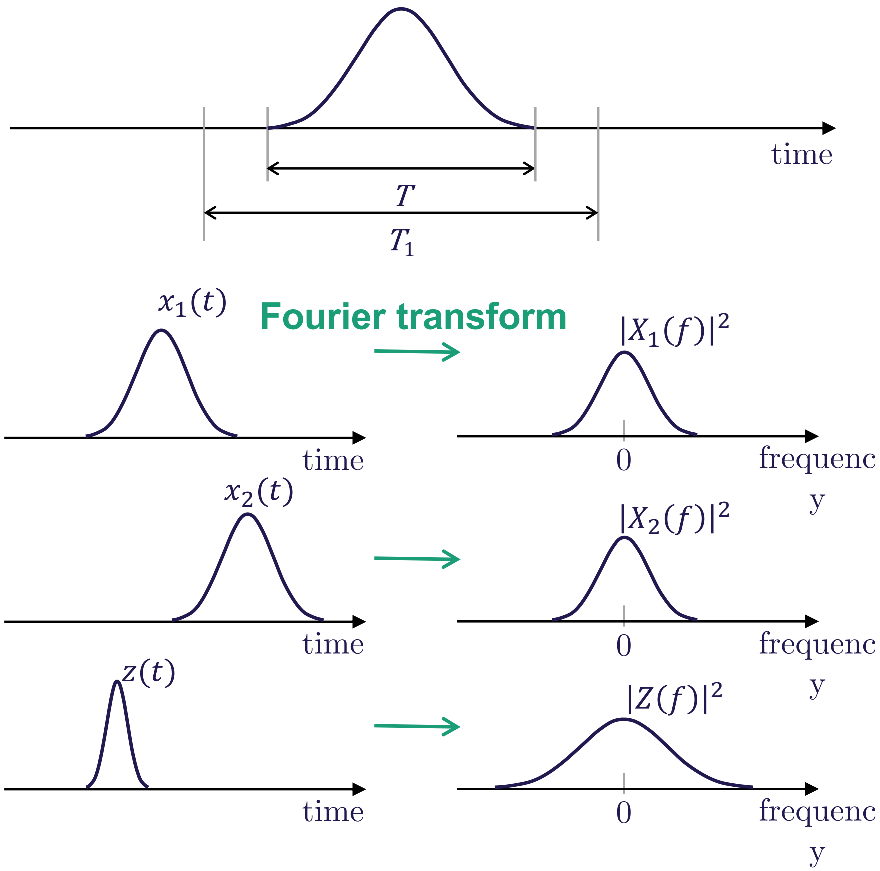
$$E_X = \int_{-\infty}^{\infty} \tilde{x}^2(t)dt = \int_{-\infty}^{\infty} |X(f)|^2 (t)\mathrm{d}f$$

How frequencies interfere with each other:

Power spectrum $S_X(f) = \lim_{T\to\infty} \frac{1}{T} E[|X_T(f)|^2]$

We can now answer:

How to **separate, in frequency,** the signals from links that are **unsynchronized in time**?



Capacity of a bandlimited channel

Frequency channel: band of contiguous frequencies, with a certain filter mask

Filter mask: provides bounds on energy radiated outside the channel

$P_{DoF} = \frac{P}{2WT}$ and $P_{DoF,N} = \frac{N_0 W}{2WT} = \frac{N_0}{2T}$ for **signal power** P, **noise power** $P_N = N_0 W$

Then, m DoFs can be treated as m c.u. of an AWGN channel

SNR is $\gamma = \frac{P_{DoF}}{P_{DoF,N}} = \frac{P}{N_0 W}$ and the **capacity** is $C^{DoF} = \frac{1}{2}\log_2\left(1 + \frac{P}{N_0 W}\right)$ [bit/DoF]

For fixed W, C^{DoF} is achieved when $T \to \infty$, $C_W = W \log_2\left(1 + \frac{P}{N_0 W}\right)$ [bps]

The spectral efficiency $\rho < C$ is obtained by normalizing by W

$$\rho = \log_2(1 + \gamma) \ [(\mathrm{bit/s})/\mathrm{Hz}]$$

Energy spent per transmission

The **energy consumption** per **complex symbol** is $E_s = \frac{PT}{WT} = \frac{P}{W}$

Then, the **SNR** becomes $\gamma = \frac{P}{N_0 W} = \frac{E_s}{N_0}$

For a given **spectral efficiency**, we have **SNR** per information bit

$$\frac{E_b}{N_0} = \frac{E_s}{\rho N_0} = \frac{\gamma}{\rho}$$

Shannon limit:

$$\rho \leq \log_2\left(1 + \frac{\rho\, E_b}{N_0}\right) \qquad \text{or} \qquad \frac{E_b}{N_0} \geq \frac{2^{\rho}-1}{\rho}$$

It tells how **efficient** a coding/modulation scheme is

Capacity of OFDM

With bandwidth W and duration T there are WT subcarriers

Simplest approach: use rectangular pulse on each subcarrier's I/Q component

$2WT$ DoFs in total

Let us observe l consecutive symbols, each of duration T

If all subcarriers are statistically identical

- Each has the same received power. The capacity is $C_W = W \log_2\left(1 + \frac{P}{N_0 W}\right)$
- **Why?** A single codeword is spread across all subcarriers with $2lWT$ channel uses

If the channel is frequency selective, a **water-filling approach** is required

- WT different codewords modulated independently, with l channel uses each

Water-filling (from chapter 8)

Assume that Xia has a limited **power** P

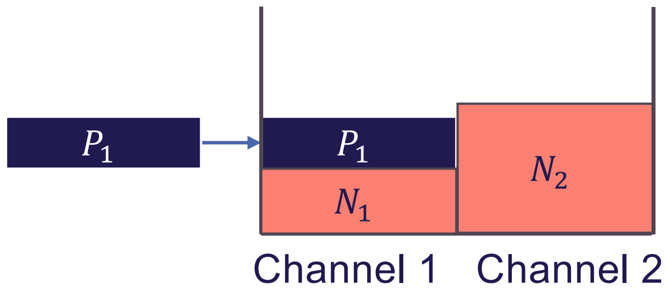
This **power** can be distributed between two frequency channels with **frequency selective noise**

$$E[|X_1|^2] + E[|X_2|^2] \leq P$$

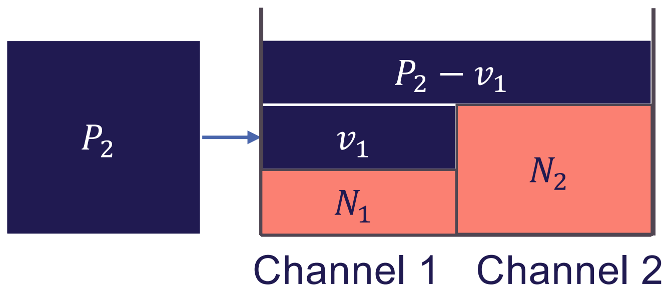
Water-willing for two channels:

1. Find the channel with the minimum noise level
$$N_1 < N_2$$
2. Fill this channel with power v_1 until
$$v_1 + N_1 = N_2$$
3. Continue allocating power to both channels equally

Total power $P_1 < v_1$



Total power $P_2 > v_1$



Multiple access and duplexing

The availability of different frequency channels enables FDMA

1. Each signal is modulated in a lower band (baseband).
2. These are mixed with the appropriate **carrier** frequency f_i.

Each transmitter or link can use a **separate channel** to **avoid interference**

✓ Removes the need for **spectrum sharing** and time-synchronization.

Coordination for channel selection and **synchronization** for frequency generation.

Full duplexing: simultaneous transmission and reception through the same channel.

Frequency Division Duplex (FDD):

Zoya and Yoshi agree on f_1 and f_2 and tune their receivers and transmitters.

Multiple access and duplexing

Neither time division nor frequency division is inherently better

TDMA

- ✓ Simple dynamic allocation of resources
- ✗ Turnaround time needed to switch between Rx and Tx mode

FDMA

- ✓ Possibility to accommodate multiple unsynchronized links
- ✗ Frequency filters are not easily adjustable to different bandwidths
- ✗ Guard bands are needed to separate carriers

Yet, **TDD** and **FDD** are fundamentally equivalent and provide same number of DOFs.

Code division and spread spectrum

Assume a symbol duration T_0 and a bandwidth $\mathrm{W}_0 = \frac{k}{T_0}$

The total number of DoF is $2T_0W_0 = 2k$

Time and **frequency** division are just two examples of **orthogonal multiplexing**

Consider now a different scenario involving 4 users

These are assigned a **code** known as **spreading sequence**

- All sequences are mutually **orthogonal**
- Users transmit their symbol several times… according to **the code**

Terminal	Code			
MD_1	1	1	1	1
MD_2	1	-1	1	-1
MD_3	1	-1	-1	1
MD_4	1	1	-1	-1

Code division and spread spectrum

The received signal for the j frequencies is

$$y_1 = b_1 + b_2 + b_3 + b_4 + n_1$$
$$y_2 = b_1 - b_2 - b_3 + b_4 + n_2$$
$$y_3 = b_1 + b_2 - b_3 - b_4 + n_3$$
$$y_4 = b_1 - b_2 + b_3 - b_4 + n_4$$

Terminal	Code			
MD_1	1	1	1	1
MD_2	1	-1	1	-1
MD_3	1	-1	-1	1
MD_4	1	1	-1	-1

To recover the message of terminal i:

Basil makes a **scalar product** of sequence i and vector $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$

$$r_i = \sum_{j=1}^{4} c_{ij} y_j = 4b_i + \sum_{j=1}^{4} c_{ij} n_j$$

The resulting **SNR** is

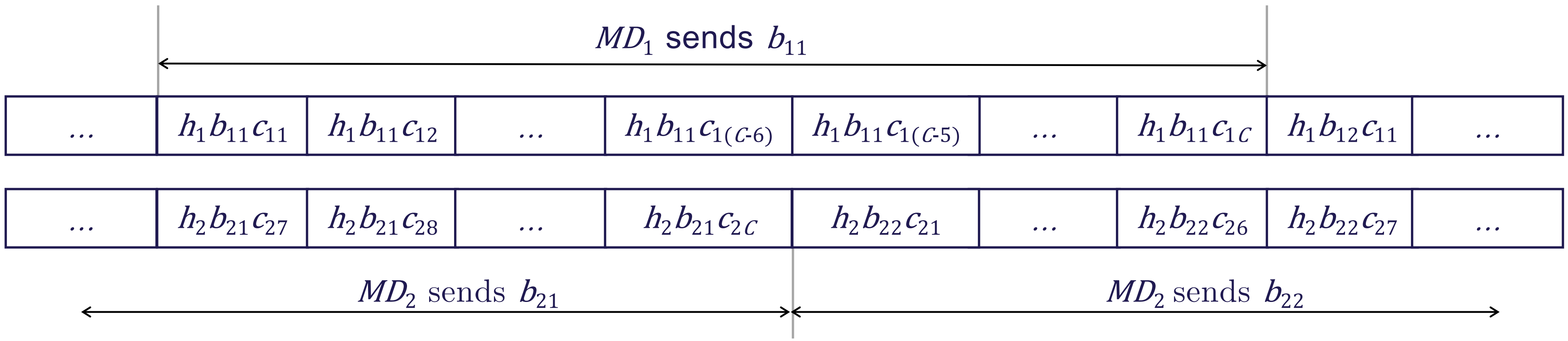
$$\gamma_i = \frac{\left|4\sqrt{P}\right|^2}{P_N + P_N + P_N + P_N} = \frac{4P}{P_N} = 4\gamma_T$$

Code division and spread spectrum

Why go through trouble of spreading?
Indeed, same performance can be achieved using time or frequency multiplexing

- One motivation could be the limit on instantaneous transmit power
- We also assumed perfect synchronization and channel knowledge

Assume now that transmissions are **chip-synchronous** but not symbol-synchronous



Code division and spread spectrum

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- We also assumed perfect synchronization and channel knowledge

Assume now that transmissions are **chip-synchronous** but not symbol-synchronous

- We look at the signals corresponding to C chips of a particular user

$$
\begin{aligned}
r_{11} &= h_1 b_{11} c_{11} + h_2 b_{21} c_{27} + n_{11} \\
r_{12} &= h_1 b_{11} c_{12} + h_2 b_{21} c_{18} + n_{12} \\
&\vdots \\
r_{1C} &= h_1 b_{11} c_{1C} + h_2 b_{22} c_{26} + n_{1C}
\end{aligned}
$$

Code division and spread spectrum

Why go through trouble of spreading? (cont.)

Again, to decode the specific message Basil uses the spreading code of that user

$$r_1 = \sum_{j=1}^{C} r_{1j}\, c_{1j} = C h_1 b_{11} + \left[h_2 b_{21} \sum_{j=1}^{C-6} c_{1j} c_{2(j+6)} + h_2 b_{22} \sum_{j=C-5}^{C} c_{1j} c_{2(j-C+6)} \right] + \sum_{j=1}^{C} c_{1j} n_{1j}$$

The term in **brackets** corresponds to the **interference**

Hence, the **cross-correlation** of sequences should be minimized

When C becomes LARGE, the **interference** resembles additional Gaussian noise

$$\textbf{SINR} = \frac{C^2 P}{C P_I + C P_N} = \frac{C P}{P_I + P_N}$$

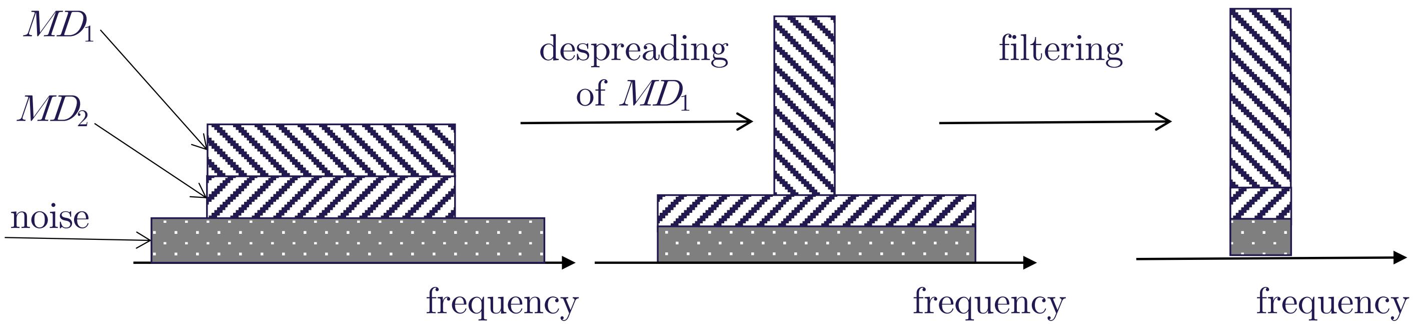
Code division and spread spectrum

In practice, there are other challenges

Lack of chip-synchronization: timing acquisition is achieved through pilots

The key in code-division multiple access (CDMA):

The bandwidth required to carry the signal is much lower that the total system bandwidth



Before de-spreading, signals resemble Gaussian noise: enables *covert communication*

Spread spectrum can also be achieved through **frequency** and/or **time** hopping

Outlook and takeaways

- **Analog waveforms** are sampled to create **discrete channel uses.**
- Multiple access in frequency: **(O)FDMA.**
- **Frequency separation** ensures decrease of interference.
- **Power spectrum** describes how frequencies would interfere with each other on average.
- Capacity of a bandlimited channel.
- Code division and spread spectrum≥